Environmental Effects on the Stress Analysis of Viscoelastic Materials

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A general systematic method is given for developing constitutive laws for engineering materials derived from actual experimental data observed in a few standard tests is given here. The constitutive theory is formulated within the framework of finite linear viscoelasticity and includes the effect of any parameter such as temperature, humidity, radiation, and strain-rate. First, the effect of a given parameter is included by analyzing the material re: ponse when the parameter is held constant at several different valves over the range of interest. Then, two methods are presented for developing a representation for arbitrary time histories of the parameter using the constant parameter analysis. Several specific examples are developed and experimental verification is included.

I. Introduction

THE development of constitutive equations is clearly one of the most fundamental issues in modern science and engineering. Constitutive equations are important because they are necessary to use the general field equations to predict accurately the response of some real material to a given stimulus. This paper brings together several disjointed constitutive theories and presents them in a single unified theory. More important, however, a systematic method is developed for deriving phenomenological constitutive equations that are based on actual experimental data taken in a standard set of experiments. The representation is for the stress-strain response of a viscoelastic solid and includes the influence of some other physical parameter; however, the method presented here could easily be applied to other constitutive processes.

Consider a typical experimental program, for example, to determine the influence of temperature on the creep of a material. Normally, a series of experiments is conducted to measure the creep in several constant-temperature environments, thereby producing a family of temperature-dependent creep curves. With this family of curves the proposed constitutive theory can predict the creep response in an arbitrary temperature environment. The first first step in developing the representation rests on the assumption that the creep curve at any one temperature can be mapped onto the creep response at any other temperature by scaling and displacing the creep curve along the ordinate and abscissa axes. The second step extends the mapping to arbitrary temperature histories. Two final constitutive representations are obtained using different physical assumptions. In one representation, hereditary effects are included while in the other they are neglected.

The environmental parameter can be any physical quantity, such as temperature, humidity, or radiation. Further, the above representation is developed for both creep and relaxation response, and the relationships between the material functions are given. The final constitutive law is, of course, for arbitrary time histories of both the mechanical and environmental parameters. The coupling, if it exists, between the mechanical and environmental parameters is neglected. For example, consider a material that is cyclically deformed. After some time a temperature rise may be observed. This temperature rise, which is attributed to dissipation, is neglected. This implies that the temporal and spatial distribution of the temperature in the body can be determined directly from the heat conduction equation and thermal boundary conditions.

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The constitutive law is developed within the framework of finite linear viscoelasticity. This is done because the method described previously can be applied to nonlinear materials with proper definition of the influence parameter. Further extension to a finite strain measure increases the applicability of the previous theory. Note that the infinitesmal theory is meaningless for a finite deformation because it does not have the invariance properties required by the principle of observer objectivity. The finite linear theory does have the proper invariance properties.

II. Basic Constitutive Relationship

Let $\sigma_{ij}(x,t)$ and $E_{ij}(x,t)$ be the components with respect to some Cartesian coordinate system of the Cauchy stress and Lagrangian strain tensors of a particle occupying position x at time t, and denote them by σ and E, respectively. Next, consider the environmental parameters that can influence the mechanical response of the material. Let $\phi(x,t)$ represent the set of all environmental parameters present at the particle at position x at time t. Assume these functions are defined and continuous for all x in the body and all of the time t in the interval $(0, \infty)$.

It is possible to obtain an integral representation for the constitutive law if the stress σ is linear in the strain history, translation invariant, nonretroactive, and continuous.† This implies that the constitutive law can be formulated within the framework of finite linear viscoelasticity[2] and written as a Riemann-Stieltjes integral. Thus assume[3]

$$\sigma(x,t) = F(t) \cdot \{ \int_{0^-}^{t} E(t-\tau) \cdot dG[\tau, \phi\overset{t}{\underset{s=0}{(s)}}] \} \cdot F^T(t) \qquad (1)$$

where $F(t)$ is the deformation gradient of the motion at position x and time t. Here G is a fourth-order tensor valved functional which; a) has the symmetry properties $G_{ijkl} = G_{jikl} = G_{ijkl}$, b) is of bounded variation on every subinterval of $[-a,\infty)$ for some $a>0$, and c) vanishes on $[-a,o)$ and is continuous on the right in $[o,\infty)$. Because the limits of integration are from 0^- to t, terms can arise from a jump discontinuity in G at $\tau=0$; i.e., the initial elastic response, are automatically included in Eq. (1).

The deformation gradient is included in Eq. (1) to account for large finite rotations. For example, consider the whip of long, thin antenna where the strains at every material point

†The properties are defined in Sec. 2 of Ref. 1.

are linear elastic, but where the accumulated displacement at the tip of the antenna can be very large. If the deformations are relatively small, then $\boldsymbol{F} \approx \boldsymbol{I}$, the identity tensor, and $\boldsymbol{E}$ becomes the infinitesimal strain measure using the displacement components. Equation (1) is then consistent with classical linear viscoelasticity. Further, it should be noted that Eq. (1) is valid only for deformation processes where the stress-induced anisotropies can be neglected. This is true because the tensorial character of Eq. (1) is fixed by the selection of the tensorial properties of $\boldsymbol{G}$. In most cases the tensorial properties of $\boldsymbol{G}$ are selected to coincide with the symmetry properties of the material in the undeformed or reference configuration.

The physical meaning of the function $\boldsymbol{G}$ in Eq. (1) can be explained by letting $\boldsymbol{E}$ have one nonvanishing component E_{11}, which is a unit step strain history applied at time $t_0=0$. The corresponding stress history predicted by Eq. (1) is

$$\sigma_{11}(t)=G_{1111}[t-t_0,\phi(\overset{t}{\underset{s=0}{s}})] \tag{2}$$

showing that $\boldsymbol{G}$ is the stress-relaxation function measured in the presence of an environmental history on the entire time interval $[0,t]$. Let us assume that the environmental history on $[0,t_0]$ when $\boldsymbol{E}=\boldsymbol{0}$ does not affect later mechanical response. Then Eq. (2) becomes

$$\begin{aligned}\sigma_{11}(t)&=G_{1111}[t-t_0\phi(\overset{t}{\underset{s=t_0}{s}})]\\&=G_{1111}[\tau,\phi(\overset{t}{\underset{s=t-\tau}{s}})]\end{aligned} \tag{3}$$

where $\tau=t-t_0$ represents the elapsed time since the strain history was applied. Substituting Eq. (3) into Eq. (1) gives

$$\sigma(x,t)=F(t)\cdot\{\int_{0^-}^{t} E(t-\tau)\cdot dG[\tau,\phi(\overset{t}{\underset{s=t-\tau}{s}})]\}\cdot F^T(t) \tag{4}$$

If the functional $\boldsymbol{G}$ has a continuous first derivative for τ in $[0,\infty)$, then Eq. (4) can be rewritten as a Riemann integral. Further, if the material is isotropic in the undeformed configuration, then $\boldsymbol{G}$ can be expanded as an isotropic functional involving two scalar material functionals and the constitutive equation for homogeneous materials becomes

$$\begin{aligned}\sigma(x,t)=F(t)\cdot\{&2G_1[0,\phi(\overset{t}{\underset{s=0}{s}})]E(t)+IG_2[0,\phi(\overset{t}{\underset{s=0}{s}})]\mathrm{tr}E\\&+\int_0^t\Big[2E(t-\tau)\frac{\partial}{\partial\tau}G_1[\tau,\phi(\overset{t}{\underset{s=t-\tau}{s}})]\\&+I\mathrm{tr}E(t-\tau)\frac{\partial}{\partial\tau}G_2[\tau,\phi(\overset{t}{\underset{s=t-\tau}{s}})]\Big]d\tau\}\cdot F^T(t)\end{aligned} \tag{5}$$

where $\boldsymbol{I}$ is the identity tensor and G_1 and G_2 are the relaxation functions in shear and dilatation, respectively.

Observe that after selecting the symmetry properties of $\boldsymbol{G}$ there is no conceptual loss of generality in considering only a one-dimensional law in place of Eq. (5). Therefore, define σ and ϵ to be a corresponding set of one-dimensional stress and strain components in shear, dilation, or tension, respectively. Then on setting $\epsilon(t)=1(t)$, the Heaviside unit step function, we can write

$$\sigma(t)=G[t,\phi(\overset{t}{\underset{s=0}{s}})] \tag{6}$$

where G is the corresponding stress relaxation functional that depends on the concurrent environmental history.

In summary, the system of constitutive equations (5) completely characterizes the mechanical response of a linear, isotropic, and homogeneous material for continuous finite deformations in the presence of an environmental history. The analysis now reduces to determining the material functions as are given by Eq. (6) from shear, dilatation, or uniaxial stress relaxation experiments. This can be accomplished using the following general approach: a) to find a convenient representation for G in Eq. (6) when the environmental history is held constant at different values for the duration of each experiment, b) to develop a representation for G using the results of part a) when the environmental history is time varying.

III. Characterization for Constant ϕ

Consider an experimental program for the determination of a scalar stress relaxation functional G that depends on a particular constant environmental history $\phi(t)=\Phi$ for all τ in $[0,t]$. A typical response $G[t,\Phi]$ to a unit step strain as given by Eq. (6) is shown in Fig. 1. Here it is assumed that the stress monotonically decreases from a defined initial modulus $G(0,\Phi)$ to a defined residual modulus $G(\infty,\Phi)$. Now let the environmental parameter be fixed at some new value ϕ_1. If the new value ϕ_1 is not too different from Φ it is reasonable to assume that the material response $G[t,\phi_1]$ will be similar to $G[t,\Phi]$. In this manner a family of relaxation curves, as shown in Fig. 1, can be obtained. The notation $G[t,\phi_p]$ signifies the dependence of the relaxation functional on the environmental history $\phi(t)=\phi_p$, which is constant throughout the body for the duration of the experiment.

Now assume the relaxation function $G[t,\Phi]$ can be mapped onto the relaxation functions $\hat{G}(t,\phi_p)$ for each value of ϕ_p. Further, assume this mapping is of the form

$$\hat{G}(t,\phi_p)=\alpha(\phi_p)I(t)+\beta(\phi_p)G[\gamma(\phi_p)t,\Phi] \tag{7}$$

where the functions α, β, and γ depend only on ϕ_p. The step function $I(t)$ is included to preserve the property that $\hat{G}(t,\phi_p)$ vanishes on the negative time interval. The quantity Φ is the reference environmental parameter for the mapping hypothesis.

The functions α, β, and γ are subject to a set of restrictions. First, is that the material response function $\hat{G}[t,\phi_p]$ must reduce to the reference response function $G[t,\Phi]$ when $\phi_p=\Phi$. This requires that

$$\alpha(\Phi)=0 \text{ and } \beta(\Phi)=\gamma(\Phi)=1 \tag{8}$$

Second, it is necessary to guarantee that the stress and strain always have the same sense. This requires that $\hat{G}[t,\phi_0]\geq 0$ for all time; thus

$$\alpha(\phi_p)+\beta(\phi_p)G[t,\Phi]\geq 0 \tag{9}$$

for all t in $(-\infty,\infty)$. Third, the sense of time must be preserved, therefore

$$\gamma(\phi_p)>0 \tag{10}$$

IV. Creep Formulation

At this point in the development it is convenient to present a constitutive formulation for viscoelastic creep. The analog to Eq. (5) for an isotropic and homogeneous material is

$$\begin{aligned}E(x,t)=F^{-1}(t)\cdot\{&2J_1[0,\phi(\overset{t}{\underset{s=0}{s}})]\sigma(t)+IJ_2[0,\phi(\overset{t}{\underset{s=0}{s}})]\mathrm{tr}\sigma\\&+\int_0^t\Big[2\sigma(t-\tau)\frac{\partial}{\partial\tau}J_1[\tau,\phi(\overset{t}{\underset{s=t-\tau}{s}})]\\&+I\mathrm{tr}\sigma(t-\tau)\frac{\partial}{\partial\tau}J_2[\tau,\phi(\overset{t}{\underset{s=t-\tau}{s}})]\Big]d\tau\}\cdot[F^{-1}(t)]^T.\end{aligned} \tag{11}$$

where J_1 and J_2 are the environmental dependent functionals in shear and dilatation, respectively. Once again letting σ and ϵ represent any of the one-dimension stress and strain components, the response to a unit step stress applied at time $t=0$ is

$$\epsilon(t)=J[t,\phi(\overset{t}{\underset{s=0}{s}})] \tag{12}$$

where J must also be determined from an experimental program as outlined for G in Sec. III.

It is convenient to introduce a creep function $\hat{J}(t,\phi_p)$ with the same structure as Eq. (7) for a constant environment history ϕ_p; that is, assume

$$\hat{J}(t,\phi_p)=\alpha^*(\phi_p)I(t)+\beta^*(\phi_p)J[\gamma^*(\phi_p)t,\Phi] \tag{13}$$

In Eq. (13), $J(t,\Phi)$ is the creep function in some reference state Φ; and α^*, β^*, γ^* are mapping functions that depend only on Φ_p.

The mapping hypothesis given by Eqs. (7) and (13) contain six mapping functions (α,β,γ and α^*, β^*,γ^*) that must be determined experimentally for constant environmental histories ϕ_p. The one-dimensional analog to Eq. (1) can be written in the form

$$\sigma(t)=\int_{0^-}^{t}\epsilon(t-\tau)d\hat{G}(\tau,\phi_p) \tag{14}$$

Since $J(t,\phi_p)$ is the strain response to stress history $\sigma(t)=I(t)$, Eq. (14) yields the following relationship between $\hat{J}$ and $\hat{G}$

$$\int_{0^-}^{t}\hat{J}(t-\tau,\phi_p)d\hat{G}(\tau,\phi_p)=I(t) \tag{15}$$

If $\hat{G}$ has been determined from an experimental program, then Eq. (15) is a Volterra integral equation for the function $\hat{J}(t,\phi_p)$. In general, the solution to Eq. (15) will not yield a material response function J in the form of Eq. (13). For certain special forms of $\hat{G}(t,\phi_p)$, however, Eq. (15) gives rise to the two following important results[3]: a) A solution to Eq. (15) can always be found in the form

$$\hat{J}(t,\phi_p)=f[\gamma(\phi_p)t,\Phi_p] \tag{16}$$

which implies that the time scaling factors for creep and relaxation are the same. b) For the special case when $\alpha=0$ in Eq. (7), the solution to Eq. (15) yields

$$\alpha^*(\phi_p)=0 \tag{17a}$$

$$\beta^*(\phi_p)=1/\beta(\phi_p) \tag{17b}$$

and

$$\gamma^*(\phi_p)=\gamma(\phi_p) \tag{17c}$$

The mapping of Eqs. (7) and (13) is motivated by experimental data. The time shift factor is the same as that introduced by Schwarzl and Staverman[4] for "thermorheologically simple" viscoelastic materials. The $\beta(\phi_p)$ term is typical of the W.L.F. equation.[5] The $\alpha(\phi_p)$ term is a more recent generalization,[3] and it does appear to increase significantly the applicability of the mapping idea.

An example of the importance of the $\alpha(\phi_p)$ term can be seen for modeling the creep response of nylon filaments in various constant temperature environments. The data given in Fig. 2 shows that increasing the temperature displaces the curve upward along the ordinate axis. Selecting 21.5°C as the reference temperature, the creep response for arbitrary temperature T can be modeled by

$$\hat{J}(t,T)=0.0356\ (T-21.5)I(t)+J[t,21.5]$$

noting that $\beta^*(T)=\gamma^*(T)=1$ and α^* is linear in T. The calculated data points are within 0.5% of the experimental data. Other places where the generalized mapping appears to work well are given in Refs. (5) and (6).

The mapping characterized by Eqs. (7) and (13) is for one environmental parameter. The development for more than

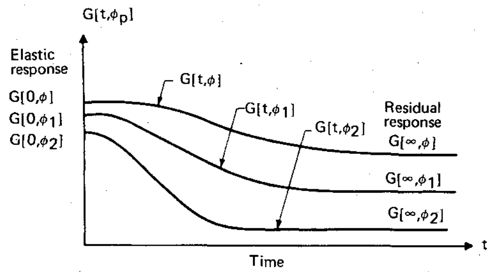

Fig. 1 Hypothetical dependence of a relaxation function on the variation of a particular environmental parameter.

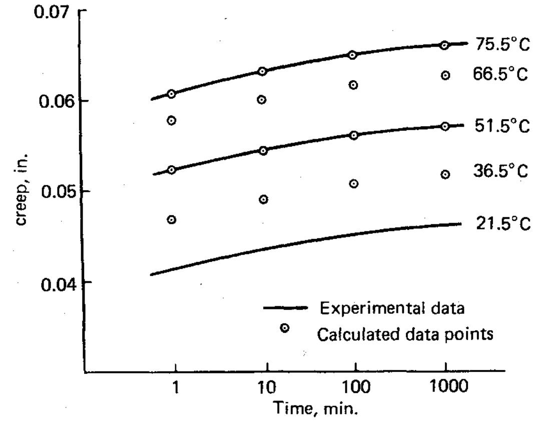

Fig. 2 Experimental and theoretical data for creep of a nylon filament at several temperatures (after Leaderman[7]).

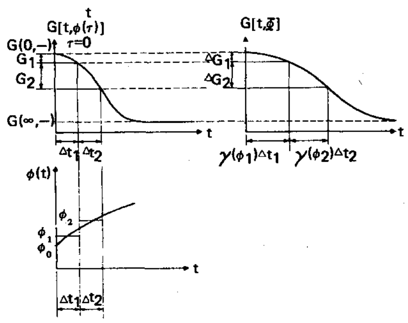

Fig. 3 Mapping of the time coordinate for $\alpha=0$ and $\beta=1$.

one parameter follows as a simple extension of the previous ideas. In a paper by Nessler and others[8] a double family of dynamic response curves for a nonlinear temperature-dependent elastomer is modeled using mapping assumption twice. The resulting representation is simple, accurate, and convenient for mathematical analysis.

V. Representation for Arbitrary $\phi(\tau)$

It is convenient to develop a theory that uses the data taken in constant parameter tests to predict the material response for arbitrary time varying environments. One expansion to transient environments rests on the assumptions that the amount of change in a relaxation function ΔG in the infitesmal interval of time $[t,t+\Delta t]$ depends only a representative value of the environmental parameter in that interval of time. The environmental history on $[0,t)$ has no influence other than to determine the stress at time t. Although the validity of this assumption must be tested expeimentally, it seems reasonable for at least some materials in slowly varying environments.

As a consequence of this assumption ΔG, which depends only on $G[t,\phi_p]$, can be utimately expressed as a function of $G[t,\Phi]$ and the mapping functions α, β, and γ. The problem reduces to finding $G[t,\phi(s)]$ for the history of s on $[0,t]$. To proceed, let the environmental history $\phi(t)$ be partitioned into N subintervals. Let Δt_i be a typical time interval $[t_{i-1},t_i]$ and let ϕ_i be a representative value of the environmental history in that interval. Denote that the value of $\phi(0)$ as ϕ_0. Then at $t=0$, Eq. (6) follows from Eq. (7) as

$$G[0,\phi(\overset{0}{\underset{s=0}{s}})]=\alpha(\phi_0)+\beta(\phi_0)G[0,\Phi]\equiv\mathring{G} \tag{18}$$

since ϕ_0 is the initial value of the environmental history.

During the first time interval $(0,\Delta t_1)$, the relaxation function will change an amount, ΔG_1, hence

$$G[\Delta t_1,\phi(\overset{\Delta t_1}{\underset{s=0}{s}})]=\mathring{G}+\Delta G_1 \tag{19}$$

The incremental change ΔG_1 is comprised of two parts. The first part, corresponding to the first term of Eq. (7), is a vertical shift in the relaxation curve due to the change in ϕ from ϕ_0 to ϕ_1. This is motivated by considering the term $\alpha(\phi)I(t)$ as an environmental dependent elastic modulus. The second part, corresponds to the second term in Eq. (7), is a monotonic decay of stress that occurs in time $(0,\Delta t_1]$ at the environmental value ϕ_1; thus

$$\Delta G_1=[\alpha(\phi_1)-\alpha(\phi_0)]$$
$$+\beta(\phi_1)\{G[\gamma(\phi_1)\Delta t_1,\Phi]-G[0,\Phi]\} \tag{20}$$

Note in Eq. (20) when $\alpha=0$ and $\beta=1$ the amount of relaxation ΔG_1 that occurs in time interval Δt_1 for the environmental state ϕ_1 will require the time interval $\gamma(\phi_1)\Delta t_1$ in the reference state Φ(see Fig. 3). Also observe that ΔG_1 could be positive or negative depending on the value of $\Delta\alpha_1$ where

$$\Delta\alpha_i=\alpha(\phi_i)-\alpha(\phi_{i-1}) \tag{21}$$

In the next time interval $(\Delta t_1,\ \Delta t_1.\Delta t_2]$, $\phi(t)$ takes on the value ϕ_2 and

$$G[\Delta t_1+\Delta t_2,\phi(\overset{\Delta t_1+\Delta t_2}{\underset{s=0}{s}})]=\mathring{G}+\Delta G_1+\Delta G_2 \tag{22}$$

where ΔG_2 depends on $\hat{G}(t,\phi_2)$. To find ΔG_2 observe that the monotonic stress relaxation at ϕ_2 during the time interval Δt_2 is $\beta(\phi_2)$ multiplied by the change in G at Φ (see Fig. 3), that is

$$\Delta G_2=\Delta\alpha_2+\beta(\phi_2)\{G[\gamma(\phi_1)\Delta t_1$$
$$+\gamma(\phi_2)\Delta t_2,\Phi]-G[\gamma(\phi_1)\Delta t_1,\Phi]\} \tag{23}$$

Hence, after some time

$$t=\sum_{i=1}^{n}\Delta t_i$$

the relaxation functional

$$G[t,\phi(\overset{t}{\underset{s=0}{s}})]=\mathring{G}+\sum_{i=1}^{n}\Delta G_i \tag{24}$$

can be written as

$$G[t,\phi(\overset{t}{\underset{s=0}{s}})]=\alpha(\phi_0)+\sum_{i=1}^{n}\Delta\alpha_i$$
$$+\beta[\phi(t)]G[\sum_{i=1}^{n}\gamma(\phi_i)\Delta t_i,\Phi]$$
$$-\sum_{i=1}^{n}G[\sum_{p=0}^{i}\gamma(\phi_p)\Delta t_p,\Phi]\{\beta(\phi_i)-\beta(\phi_{i-1})\} \tag{25}$$

after using Eqs. (20) and (23) in Eq. (24) and generalizing for ΔG_i. Letting $N\rightarrow\infty$ for a fixed time interval $[0,t]$, Eq. (25) can be written as

$$G[t,\phi(\overset{t}{\underset{s=0}{s}})]=\alpha[\phi(t)]+\beta[\phi(t)]G[\int_0^t\gamma[\phi(\tau)]d\tau,\Phi]$$
$$-\int_{0+}^{t}G[\int_{0+}^{\tau}\gamma[\phi(\theta)]d\theta,\Phi]d\beta[\phi(\tau)] \tag{26}$$

The representation in Eq. (26) is consistent with the results of Morland and Lee[9] for thermorheologically simple materials. In this case $\alpha=0$ and $\beta=1$, then Eq. (26) becomes

$$G[t,\phi(\overset{t}{\underset{s=0}{s}})]=G[\xi,\Phi] \tag{27}$$

where

$$\xi=\int_0^t\gamma[\phi(\theta)]d\theta \tag{28}$$

In Eq. (28) ξ is the pseudo-time introduced by Morland and Lee and implies that the change in the rate of relaxation due to ϕ can be accounted for by a one-to-one mapping of the time coordinate.

To develop a general representation for G, first note for spatially variant environmental fields $\phi(t,x)$, the mapping functions α,β, and γ become functions of position and time. Second, the material response functionals defined in Eqs. (3) and (5) are only for the environmental history in the time interval $[t_0,t]$. Thus, a material functional for time and spatially varying environmental parameters compatible with Eq. (5) must be of the form

$$G[t-t_0,\phi(\overset{t}{\underset{s=t_0}{s}},x)]=\alpha[\phi(t,x)]$$
$$+\beta[\phi(t,x)]G[\int_{t_0}^{t}\gamma[\phi(\tau',x)]d\tau',\Phi]$$
$$-\int_{t_0}^{t}G[\int_{t_0}^{\tau}\gamma[\phi(\theta,x)]d\theta,\Phi]d\beta[\phi(\tau,x)] \tag{29a}$$

If the creep response is considered, a similar development gives the scalar creep function as

$$J[t-t_0,\phi(\overset{t}{\underset{s=t_0}{s}},x)]=\alpha^*[\phi(t,x)]$$
$$+\beta^*[\phi(t,x)]J[\int_{t_0}^{t}\gamma^*[\phi(\tau',x)]d\tau',\Phi]$$
$$-\int_{t_0}^{t}J[\int_{t_0}^{\tau}\gamma^*[\phi(\theta,x)]d\theta,\Phi]d\beta^*[\phi(\tau,x)] \tag{29b}$$

To conclude, recall that the mapping functions α, β, and γ, and the material response function in the reference state $G[t,\Phi]$ are determinable from an experimental program. The environmental history $\phi(t,x)$ can be found from the appropriate physical law and boundary conditions or measured experimentally. Eq. (29) can then be evaluated for the material response functional G and then used with Eq. (5) for stress analysis and/or design purposes.

It is appropriate to recall that these results are formulated within the framework of finite linear viscoelasticity, as evidenced by the single integral constitutive Eq. (1). This assumption implies the additivity of response which places a restriction on the memory characteristics of the material. The assumption is correct for some materials but certainly is not valid for all materials. Propellants, for example, have permanent rather than fading memory characteristics, and they do not obey the additivity assumption of the Boltzman integral. In the last section of this paper modification to the constitutive theory is presented that permits extension to some types of nonlinear viscoelastic response.

VI. Hereditary Development for Arbitrary $\phi(\tau)$

The pseudo-time theory introduced by Morland and Lee in 1960 is widely accepted theoretically, but has not had as much success experimentally. In the more recent development[10] previously given, the pseudo-time theory is extended to include vertical shifting and scaling; however, the development is based on the same physical assumptions. Thus, the same inherent experimental difficulty may be present in both theories.

In Sec. VI an alternative approach is given that includes a much different environmental history dependence. Recall that the previous theory was developed on the assumption that the amount of change in a material function in any given increment of time depends only on a representative value of the environment in that increment of time. Here that restriction is abandoned so that the material response can depend on the previous environmental history in a hereditary manner. That is, in classical linear viscoelasticity the Boltzman superposition principle gives the stress as a hereditary integral of the strain for a constant environment parameter, i.e., say, temperature. Here the roles of one mechanical parameter and one environmental parameter are interchanged so that, a) the stress relaxation is given as a hereditary function of the temperature at constant strain, or b) the creep is given as a hereditary function of the temperature at constant stress. Thus, the hereditary law arises out of a natural extension of the Boltzman superposition principle.

Define for convenience a viscoelastic creep function $\hat{J}[t,\phi_p]$ as the strain response to a unit step stress in the constant environment ϕ_p. (The function $\hat{J}$ may have a representation similar to Eq. (13) but this assumption is not necessary for the current development.) The response of the material to a creep and recovery test in the environment ϕ_p can now be developed. Let the stress history be given by

$$\sigma(t)=I(t)-I(t-t_1) \tag{30}$$

for some $t_1>0$. The strain for any time t is then given by

$$\epsilon(t)=\hat{J}[t,\phi_p]-\hat{J}[t-t_1,\phi_p] \tag{31}$$

For the development that follows, assume $\hat{J}[t,\phi_p]$ is a smooth function of t and ϕ_p, and the environmental parameter is a smooth function of t.

The response of the material in a time-varying environment can now be developed using the concept embodied in Eqs. (30) and (31). Let the environmental history be partitioned in N subintervals. Denote the time at the beginning of each interval by t_i $(i=0,\ 1,\ 2....,n-1)$ with $t_0=0$ and $t_N=t$. Let $\phi_i=\phi(t_i^+)$ be a representative value of the environment in each interval. Apply a unit step stress history in the form

$$\sigma(t)=I(t)=I(t-t_0)-I(t-t_1)+I(t-t_1)$$
$$-I(t-t_2)+I(t-t_2)+.... \tag{32}$$

Parallel to the response in Eq. (31), assume the response to Eq. (32) is given by

$$\epsilon(t)=\hat{J}(t-t_0,\phi_0)-\hat{J}(t-t_1,\phi_0)+\hat{J}(t-t_i,\phi_1)+.... \tag{33}$$

Rearranging terms and defining $\Delta\phi_i=\phi_i-\phi_{i-1}$ allows Eq. (33) to be rewritten as

$$\epsilon(t)=\hat{J}[t,\phi(0)]$$
$$+\sum_{i=1}^{N-1}\frac{\hat{J}[t-t_i,\phi_i]-\hat{J}[t-t_i,\phi_{i-1}]}{\Delta\phi_i}\Delta\phi_i \tag{34}$$

Letting $N\rightarrow\infty$ for a fixed interval $[0,t]$ and using the definition of a partial derivative Eq. (34) becomes

$$\epsilon(t)=\hat{J}[t,\phi(0)]$$
$$+\int_0^t\frac{\partial\hat{J}[t-\tau,\phi(\tau)]}{\partial\phi(\tau)}\frac{d\phi(\tau)}{d\tau}d\tau \tag{35}$$

Note that $\hat{J}[t,\phi(0)]$ is the creep response at time t due to the initial environment. The integral is the accumulated change in response, due to changing the environment.

An alternative form for Eq. (35) can be derived by recombining terms in Eq. (33) and forming a different sum. The same expression can also be obtained by noting that

$$[d\hat{J}[t-\tau,\phi(\tau)]/d\tau]=-[\partial\hat{J}[t-\tau,\phi(\tau)]/\partial(t-\tau)]$$
$$+[\partial\hat{J}[t-\tau,\phi(\tau)]/\partial\phi(\tau)]\ [d\phi(\tau)/d\tau] \tag{36}$$

Then on using Eq. (36) in Eq. (35), it follows that

$$\epsilon(t)=\hat{J}[0,\phi(t)]+\int_0^t\frac{\partial\hat{J}[t-\tau,\phi(\tau)]}{\partial(t-\tau)}d\tau \tag{37}$$

Finally, for completeness, Eqs. (35) and (37) can be put in a form compatible with Eq. (11) and including spatial variation in $\phi(t,x)$. Noting that $\epsilon(t)$ in Eqs. (35) and (37) is indeed the creep for an arbitrary environmental history, then

$$J[t-t_0,\phi(\overset{t}{\underset{s=t_0}{s}},x)]=J[t-t_0,\phi(t_0,x)]$$
$$+\int_{t_0}^t\frac{\partial\hat{J}[t-\tau,\phi(\tau,x)]}{\partial\phi(\tau,x)}\frac{\partial\phi(\tau,x)}{\partial\tau}d\tau$$
$$=\hat{J}[t-t_0,\phi(t,x)]+\int_0^t\frac{\partial\hat{J}[t-\tau,\phi(\tau,x)]}{\partial(t-\tau)}d\tau \tag{38}$$

The same closing comments apply to Eq. (38) as did to Eq. (29) since the material functions and environmental parameter history are determinable from standard experiments and/or calculations.

VII. Extension to Nonlinear Materials

The ideas and techniques developed in the earlier sections of this paper can easily be amended to include nonlinear viscoelastic materials.[11] In this case it is necessary to identify ϕ with an appropriate measure of stress or strain. Let us assume once again that we are concerned with a homogeneous and isotropic material. Then the one-dimensional creep function $J(t,\phi)$ and stress parameter ϕ can be associated as given in Table 1.

To show how the nonlinearity enters into the material response, consider a series of one-dimensional creep experiments. Let the stress history be given by

$$\sigma(t)=\phi_p I(t) \tag{39}$$

where ϕ_p $(p=1,2,...)$ are constant. The creep response function $\hat{J}$ is defined as

$$\epsilon(t,\phi_p)/\phi_p=\hat{J}(t,\phi_p)=\alpha^*(\phi_p)I(t)+\beta^*(\phi_p)J[t,\Phi] \tag{40}$$

Table 1 Creep function and stress parameter relationship

Creep function $J(t,\phi)$	Definition of stress parameter ϕ
uniaxial	uniaxial
dilatation	hydrostatic
shear	octrahedral shear

where $\hat{J}$ depends on both time and stress ϕ_p. If the material is linear then $\hat{J}$ is a function of time alone. Further, the right-hand side of Eq. (40) assumes a particular representation for $\hat{J}$ as in Eq. (13) except that the time shift factor $\gamma^*(\phi_p)=1$.

Experience has shown that the two functions α^* and β^* are adequate for a large class of materials and the omission of γ^* leads to a very convenient analytical representation for arbitrary stress histories. The function $J[t,\Phi]$ is the creep response for some reference value $\phi_p=\Phi$ and the scaling functions must satisfy

$$\alpha^*(\Phi)=0, \quad \beta^*(\Phi)=1 \tag{41}$$

so that

$$\hat{J}(t,\Phi)=J(t,\Phi)$$

The extension to time-varying stress histories can be obtained by partitioning the stress into N subintervals. Let t_i, $(i=0,1,...N-1)$ be the time at the beginning of $i+1$ interval. Designate the stress at the beginning of each interval by $\phi(t_i)=\phi_i$. Then the stress history can be approximated by

$$\phi(t)\cong\sum_{i=0}^{N-1}\phi_i[I(t-t_i)-I(t-t_{i+1})]+\phi(t)I(0), \tag{42}$$

with Eq. (40) the strain response to Eq. (42) can be written as

$$\epsilon(t)=\sum_{i=0}^{N-1}\phi_i[\hat{J}(t-t_i,\phi_i)-\hat{J}(t-t_{i+1},\phi_i)]+\phi(t)\hat{J}(0,\phi_i) \tag{43}$$

Substituting the right hand side of Eq. (40) into Eq. (43) gives

$$\epsilon(t)=\sum_{i=0}^{N-1}\phi_i\alpha^*(\phi_i)[I(t-t_i)-I(t-t_{i+1})]$$
$$+\sum_{i=0}^{N=1}\phi_i\beta^*(\phi_i)[J(t-t_i,\phi)-J(t-t_{i+1},\phi)]$$
$$+\phi(t)\{\alpha^*[\phi(t)]+\beta^*[\phi(t)]J[0,\Phi]\} \tag{44}$$

Observe that the first summation vanishes. Let $N\rightarrow\infty$ for a fixed time interval, then Eq. (44) can be written as the Stielljes integral

$$\epsilon(t)=\sigma(t)\ \{\alpha^*(t)+\beta^*(t)J(0)\}$$
$$+\int_{0^+}^{t}\sigma(\tau)\beta^*(\tau)dJ(t-\tau) \tag{45}$$

using the notation

$$\phi(t)=\sigma(t), \quad J(t,\Phi)=J(t), \tag{46a}$$

$$\alpha^*(t)=\alpha^*[\phi(t)], \quad \beta^*(t)=\beta^*[\phi,(t)], \tag{46b}$$

Expanding into a Riemann integral and letting $t-\tau=\xi$, the aforementioned becomes

$$\epsilon(t)=\sigma(t)\ [\alpha^*(t)+\beta^*(t)J(0)]$$
$$+\int_0^t\sigma(t-\xi)\beta^*(t-\xi)\frac{dJ(\xi)}{d\xi}d\xi \tag{47}$$

Equation (47) reduces to the classical linear law for $\alpha^*=0$ and $\beta^*=1$, as required by Eq. (40). The previous representation can be rewritten using a Riemann convolution integral

$$\epsilon(t)=\sigma(t)\ [\alpha^*(t)+\beta^*(t)J(0)]$$
$$+\int_0^t f(t-\xi)\frac{dJ(\xi)}{d\xi}d\xi \tag{48}$$

where

$$f(t-\xi)=\sigma(t-\xi)\beta^*(t-\xi) \tag{49}$$

Thus, all the properties of Riemann convolutions can be used for nonlinear viscoelasticity.

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